VII.2 Algebraic Künneth formula and Universal Coefficient theorem for homology

Algebraic Künneth formula

9. (Tensor product of chain complexes)
$\mathcal{C}, \mathcal{C}'$: chain complexes $\Rightarrow \mathcal{C} \otimes \mathcal{C}'$:chain complex :

Define $(\mathcal{C} \otimes \mathcal{C}')_n := \bigoplus_{p=0}^{n} (C_p \otimes C'_{n-p})$ and $\partial : (\mathcal{C} \otimes \mathcal{C}')_n \to (\mathcal{C} \otimes \mathcal{C}')_{n-1}$ by
$\partial(c \otimes c') = \partial c \otimes c' + (-1)^p c \otimes \partial c'$ for $\forall c \otimes c' \in C_p \otimes C'_{n-p}$
$\Rightarrow \partial^2 = 0$: clear. ($\because (-1)^{p-1} \partial c \otimes \partial c' + (-1)^p \partial c \otimes \partial c' = 0$)

Want to compare $H(\mathcal{C} \otimes \mathcal{C}')$ and $H(\mathcal{C}) \otimes H(\mathcal{C}')$:
We have a well-defined canonical homomorphism $i : H(\mathcal{C}) \otimes H(\mathcal{C}') \to H(\mathcal{C} \otimes \mathcal{C}')$ given by $H_p(\mathcal{C}) \otimes H_{n-p}(\mathcal{C}') \to H_n(\mathcal{C} \otimes \mathcal{C}')$.

$$\{z\} \otimes \{z'\} \mapsto \{z \otimes z'\}$$

Check this is well-defined : (1) z, z': cycles $\Rightarrow z \otimes z'$ is a cycle. (clear)
(2) $(z + \partial c) \otimes z' = z \otimes z' + (\partial c \otimes z') = z \otimes z' + \partial(c \otimes z')$

10. Assume R : PID , $\mathcal{C}$: free
Show algebraic Künneth :

Split chain complex $\mathcal{C}$ into two short exact sequences as usual :
(1)$0 \to Z \xrightarrow{i} \mathcal{C} \xrightarrow{\partial} \bar{B} \to 0$ ($\bar{B}_p := B_{p-1}$)
(2)$0 \to B \xrightarrow{j} Z \to H(\mathcal{C}) \to 0$

Start with tensoring (1) with $\mathcal{C}'$ to compute $H(\mathcal{C} \otimes \mathcal{C}')$.
Note $\bar{B} \subset \mathcal{C}$ is free and (1) is a splitting s.e.s.
(1)$\Rightarrow 0 \to Z \otimes \mathcal{C}' \xrightarrow{i \otimes 1} \mathcal{C} \otimes \mathcal{C}' \xrightarrow{\partial \otimes 1} \bar{B} \otimes \mathcal{C}' \to 0$: s.e.s.
(i.e., $0 \to Z_p \otimes C'_{n-p} \to C_p \otimes C'_{n-p} \to \bar{B}_p \otimes C'_{n-p} \to 0$: s.e.s.)

Here we view Z as a chain complex : $\to Z_p \xrightarrow{\partial=0} Z_{p-1} \xrightarrow{\partial=0} \cdots$ and similarly for $\bar{B}$.

Remark. $f : \mathcal{C} \to \mathcal{C}'$ and $g : \mathcal{D} \to \mathcal{D}'$: chain map
$\Rightarrow f \otimes g : \mathcal{C} \otimes \mathcal{D} \to \mathcal{C}' \otimes \mathcal{D}'$: chain map

$$c \otimes d \mapsto f(c) \otimes g(d)$$
$$(\partial c \otimes d + (-1)^p c \otimes \partial d \mapsto f(\partial c) \otimes g(d) + (-1)^p f(c) \otimes g(\partial d)$$
$$= \partial f(c) \otimes g(d) + (-1)^p f(c) \otimes \partial g(d))$$

Now Snake lemma $\Longrightarrow$
$\cdots \to H_n(Z \otimes \mathcal{C}') \to H_n(\mathcal{C} \otimes \mathcal{C}') \to H_n(\bar{B} \otimes \mathcal{C}') \to H_{n-1}(Z \otimes \mathcal{C}') \to \cdots$

Show $H(Z \otimes \mathcal{C}') = Z \otimes H(\mathcal{C}')$ and $H(\bar{B} \otimes \mathcal{C}') = \bar{B} \otimes H(\mathcal{C}')$:

$$\to Z_p \overset{\partial=0}{\to} Z_{p-1} \overset{\partial=0}{\to} \cdots$$
$$\Rightarrow \to (Z \otimes \mathcal{C}')_n \overset{\partial}{\to} (Z \otimes \mathcal{C}')_{n-1} \to \cdots \text{ is given by}$$
$$\begin{array}{ccc} \cdots \to Z_p \otimes C'_{n-p} & \overset{"\partial"=(-1)^p 1\otimes\partial}{\longrightarrow} & Z_p \otimes C'_{n-p-1} \to \cdots \\ z \otimes c' & \mapsto & (-1)^p z \otimes \partial c' \end{array}$$
$$= Z_p \otimes (\cdots \to C'_{n-p} \overset{\partial'=(-1)^p\partial}{\longrightarrow} C'_{n-p-1} \to \cdots)$$
$$= \left\{ \begin{array}{c} Z_p \otimes (1)'_{n-p} \\ Z_p \otimes (2)'_{n-p} \end{array} \right\} \text{(Since } Z_p \text{ is free, } Z_p\otimes \text{ preserves s.e.s.)}$$
$$\begin{array}{l} \Rightarrow H_n(Z \otimes \mathcal{C}') = \bigoplus_{p=0}^{n} H(Z_p \otimes \mathcal{C}'_{n-p}) \cong \bigoplus_{p=0}^{n} Z_p \otimes H_{n-p}(\mathcal{C}') = (Z \otimes H(\mathcal{C}'))_n \\ \qquad \{z \otimes z'\} \qquad\qquad\qquad \mapsto \qquad\qquad\qquad z \otimes \{z'\} \end{array}$$

Similarly, $H(\bar{B} \otimes \mathcal{C}') = \bar{B} \otimes H(\mathcal{C}')$
$\{b \otimes z'\} \leftrightarrow b \otimes \{z'\}$
$$\begin{array}{l} \therefore \cdots \overset{\phi_n}{\to} (Z \otimes H(\mathcal{C}'))_n \to H_n(\mathcal{C} \otimes \mathcal{C}') \to (\bar{B} \otimes H(\mathcal{C}'))_n \overset{\phi_{n-1}}{\to} \cdots \\ \qquad z \otimes \{z'\} \mapsto \{z \otimes z'\} \qquad\qquad (= (B \otimes H(\mathcal{C}'))_{n-1}) \end{array}$$

$\Rightarrow 0 \to cok\phi_n \to H_n(\mathcal{C} \otimes \mathcal{C}') \to ker\phi_{n-1} \to 0,$
where $\phi_n : (B \otimes H(\mathcal{C}'))_n \to (Z \otimes H(\mathcal{C}'))_n$

Now, (2)$\Rightarrow$ $0 \to B_p \overset{j}{\to} Z_p \to H_p(\mathcal{C}) \to 0 :$ s.e.s.
$\underset{Z_p:free}{\overset{\otimes H_{n-p}(\mathcal{C}')}{\Longrightarrow}}$ $0 \to Tor(H_p(\mathcal{C}'), H_{n-p}(\mathcal{C}')) \to B_p \otimes H_{n-p}(\mathcal{C}') \to Z_p \otimes H_{n-p}(\mathcal{C}') \to H_p(\mathcal{C}) \otimes H_{n-p}(\mathcal{C}') \to 0$

$$\therefore cok\phi_n = \bigoplus_{p=0}^{n} H_p(\mathcal{C}) \otimes H_{n-p}(\mathcal{C}') \text{ and } ker\phi_{n-1} = \bigoplus_{p=0}^{n-1} Tor(H_p(\mathcal{C}), H_{n-p-1}(\mathcal{C}'))$$

$$\begin{array}{l} \therefore 0 \to \bigoplus_{p=0}^{n} H_p(\mathcal{C}) \otimes H_{n-p}(\mathcal{C}') \to H_n(\mathcal{C} \otimes \mathcal{C}') \to \bigoplus_{p=0}^{n-1} Tor(H_p(\mathcal{C}), H_{n-p-1}(\mathcal{C}')) \to 0 \\ \qquad\qquad \{z\} \otimes \{z'\} \quad \mapsto \quad \{z \otimes z'\} \end{array}$$
or $0 \to (H(\mathcal{C}) \otimes H(\mathcal{C}'))_n \to H_n(\mathcal{C} \otimes \mathcal{C}') \to (H(\mathcal{C}) * H(\mathcal{C}'))_{n-1} \to 0.$

Sequence splits if $\mathcal{C}$ and $\mathcal{C}'$ are free : Since $\bar{B}$ is free,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \underset{i}{\overset{\exists\pi}{\rightleftarrows}} & \mathcal{C} & \longrightarrow & \bar{B} & \longrightarrow & 0 \\ & & \downarrow p & \swarrow_{\psi = p\cdot\pi} & & & & & \\ & & H(\mathcal{C}) & & & & & & \end{array} \quad \Rightarrow \quad \begin{array}{c} \mathcal{C}\otimes\mathcal{C}' \xrightarrow{\psi\otimes\psi'} H(\mathcal{C})\otimes H(\mathcal{C}') \\ c\otimes c' \mapsto \{\pi(c)\}\otimes\{\pi'(c')\} \end{array}$$

,where again $H(\mathcal{C})$ is a trivial chain complex.
$\Rightarrow H(\mathcal{C}\otimes\mathcal{C}') \overset{(\psi\otimes\psi')*}{\longrightarrow} H(H(\mathcal{C})\otimes H(\mathcal{C}')) = H(\mathcal{C})\otimes H(\mathcal{C}')$

Naturality follows as before. $\square$

11. Universal Coefficient Theorem for Homology

In algebraic Künneth, let $\mathcal{C}'_n = \left\{ \begin{array}{ll} G & ,n=0 \\ 0 & ,otherwise \end{array} \right\}$, where G is R(PID)-mod.

Then $H_q(\mathcal{C}') = \left\{ \begin{array}{ll} G & ,q=0 \\ 0 & ,q\neq 0 \end{array} \right\}$ and $(\mathcal{C}\otimes\mathcal{C}')_n = C_n\otimes G$. Hence we have

U.C.T. : If $\mathcal{C}$ is free, then $0\to H_n(\mathcal{C})\otimes G\to H_n(\mathcal{C}\otimes G)\to Tor(H_{n-1}(\mathcal{C}),G)\to 0$ which splits (not canonical), where $H_n(\mathcal{C}\otimes G) = H_n(\mathcal{C};G)$ homology with coefficient G.

Note. For a proof for splitting, see the proof of algebraic Künneth :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z' & \overset{i}{\longrightarrow} & \mathcal{C}' & \longrightarrow & \bar{B}' & \longrightarrow & 0 \\ & & \downarrow p & \swarrow_{Want\ \psi = p\cdot\pi} & & & & & \\ & & H(\mathcal{C}') & & & & & & \end{array}$$

but $H_p(\mathcal{C}') = Z'_p = 0$ if $p\neq 0$ and for $p=0, Z'_0 = \mathcal{C}'_0 = H_0(\mathcal{C}') = G$.

Note 1. In particular, if $\mathcal{C}$ is a chain complex of free abelian group and R, $\forall$ commutative ring with 1 (so abelian group), then $0\to H_n(\mathcal{C})\otimes R\to H_n(\mathcal{C}\otimes R)\to H_{n-1}(\mathcal{C})*R\to 0$, where $\otimes = \underset{\mathbb{Z}}{\otimes}$, i.e., abelian group tensor product and $* = \underset{\mathbb{Z}}{*}$,i.e., abelian group torsion product and hence if $\mathcal{C} = S_n(X)$ (or $S_n(X,A)$), then $0\to H_n(X;\mathbb{Z})\otimes R\to H_n(X;R)\to Tor(H_{n-1}(X;\mathbb{Z}),R)\to 0$

Note 2. A: abelian group and R : commutative ring with 1.
$\Rightarrow A\otimes R$ has a canonical R-module structure given by $r(a\otimes x) = a\otimes(rx)$
R-module $S_n(X;R)$ defined earlier is exactly $S_n(X)\otimes R$.

숙제 28. Compute $H(P^n;\mathbb{Z}/2)$ using $H(P^n;\mathbb{Z})$ and compare. (Use U.C.T.)

숙제 29. R : PID $\Rightarrow \chi(X) = \chi(X;R)$ (Use U.C.T.)

Note 3. If R is a field, then every R-module is free. Hence $H(\mathcal{C}) \otimes H(\mathcal{C}') \cong H(\mathcal{C} \otimes \mathcal{C}')$ (or $H_n(\mathcal{C} \otimes \mathcal{C}') \cong \bigoplus_{p=0}^{n} H_p(\mathcal{C}) \otimes H_{n-p}(\mathcal{C}')$) and $H_n(\mathcal{C}) \otimes G \cong H_n(\mathcal{C} \otimes G)$, $\forall$ vector space G.

Note (4) Let R be a field with ch(R)=0.
$\Rightarrow H_n(X;\mathbb{Z}) \bigotimes R \cong H_n(X;R)$, since R is a torsion free abelian group and hence $\mathrm{Tor}(H_{n-1}(X;\mathbb{Z}), R) = 0$.

Proposition Let B be a torsion free abelian group. Then $B * A = 0$.
Sketch of proof
Note that any R-module is a direct limit of its finitely generated submodule.
And finitely generated torsion free is free if R is a P.I.D.(Structure theorem).
Now tensor product commutes with direct limit(easy exercise).
$\Rightarrow *$ commutes with direct limit.
$\Rightarrow \mathrm{Tor}(C, A) = 0$ if C is a torsion free R-module with R: P.I.D.

□

12. Eilenberg-Zilber Theorem
$S(X \times Y)$ and $S(X) \bigotimes S(Y)$ are naturally chain homotopy equivalent.
Hence $H(X \times Y) \cong H(S(X) \bigotimes S(Y))$.

Proof Use Acyclic Model Theorem. Recall AMT.

> **Acyclic Model Theorem**
> Let $F, F' : \mathcal{T} \to \mathcal{C}$ be functors and $\mathcal{M} \subset Ob(\mathcal{T})$, where $\mathcal{C}$ is a category of chain complexes.
> Suppose
>
> (1) F' is acyclic relative to $\mathcal{M}$, i.e., $F'(M)$ is acyclic $\forall M \in \mathcal{M}$.
>
> (2) F is free relative to $\mathcal{M}$, i.e., $\forall p, \exists$indexed family$\{M_\alpha\}_{\alpha \in J_p}and\{i_\alpha\}_{\alpha \in J_p}, M_\alpha \in \mathcal{M}, i_\alpha \in F_p(M_\alpha)$ such that the indexed family $\{F(\sigma) i_\alpha\}_{\alpha \in J_p}, \sigma \in \hom(M_\alpha, X)$ is a basis for $F_p(X)$.
> Then
>
> (1) $\exists$ a natural transformation $\tau : F \to F'$ which induces a given natural transformation $\tau_0 : H_0(F) \to H_0(F')$.
>
> (2) Given two such natural transformations $\tau, \tau' : F \to F'$ with $\tau_0 = \tau_0', \tau \simeq \tau'$

Eilenberg-Zilber Situation
Let $\mathcal{T}$ be the category of pairs (X, Y) of topological spaces.
Consider $F : \mathcal{T} \to \mathcal{C}, F(X, Y) = S(X \times Y)$ and $F' : \mathcal{T} \to \mathcal{C}, F'(X, Y) = S(X) \bigotimes S(Y)$.
$\Rightarrow$ These are clearly functors :

$$\begin{array}{ccccc} (X,Y) & \longrightarrow & S(X \times Y) & \qquad & S(X) \bigotimes S(Y) \\ {\scriptstyle (f,g)}\downarrow & & \downarrow{\scriptstyle F(f,g)=(f\times g)_\sharp} & & \downarrow{\scriptstyle F'(f,g)=f_\sharp \otimes g_\sharp = \bigoplus_{p=0}^{n} f_{\sharp p} \otimes g_{\sharp n-p}} \\ (X', Y') & \rightarrow & S(X' \times Y') & & S(X') \bigotimes S(Y') \end{array}$$

Let $\mathcal{M} = \{(\triangle^p, \triangle^q),\ p, q \geq 0\}$, where $\triangle^p$ is a standard p-simplex.
(1) F, F' are both acyclic relative to $\mathcal{M}$:
That F is acyclic relative to $\mathcal{M}$ is clear since $\triangle^p \times \triangle^q$ is contractible.

Consider F'.
$H_n(S(X) \bigotimes S(Y)) \cong \bigoplus_{p=0}^{n} H_p(X) \bigotimes H_{n-p}(Y) \bigoplus \bigoplus_{p=0}^{n-1} \mathrm{Tor}(H_p(X), H_{n-p-1}(Y))$에서 $X = \triangle^p, Y = \triangle^q$라고 두자. 그러면 $\widetilde{H}(\triangle^p) = 0 = \widetilde{H}(\triangle^q)$이고
$H_0(S(X) \bigotimes S(Y)) \cong H_0(X) \bigotimes H_0(Y) \cong R \bigotimes R = R$ 이며 $n > 0$인 n에 대해서는 $H_n(S(X) \bigotimes S(Y)) = 0$이다.

(2) F is free relative to $\mathcal{M}$:

For each n, choose $(\triangle^n, \triangle^n) \in \mathcal{M}$ and $d_n \in S_n(\triangle^n \times \triangle^n)$, where $d_n : \triangle^n \to \triangle^n \times \triangle^n$ is the diagonal map : $t \mapsto (t,t)$.
Now for each $(f,g) \in \hom((\triangle^n, \triangle^n),(X,Y)), \{F(f,g)d_n\}$ form a basis for $S_n(X \times Y)$ since

$$\begin{array}{ccc} & \overset{\sigma_Y}{\nearrow} & Y \\ & & \uparrow P_Y \\ \forall \sigma : \triangle^n & \longrightarrow & X \times Y \\ & & \downarrow P_X \\ & \underset{\sigma_X}{\searrow} & X \end{array}$$

can be written uniquely as $\sigma = (\sigma_X, \sigma_Y) = (\sigma_X \times \sigma_Y) \circ d_n$.

(3) F' is free relative to $\mathcal{M}$:
For each n, choose $(\triangle^p, \triangle^q) \in \mathcal{M}$ with $p+q=n$ and $i_p \otimes i_q \in (S(\triangle^p) \bigotimes S(\triangle^q))_n = \bigoplus_{i+j=n} S_i(\triangle^p) \bigotimes S_j(\triangle^q)$, where $i_p = id. : \triangle^p \to \triangle^p$.
For each $(\sigma, \tau) \in \hom((\triangle^p, \triangle^q),(X,Y))$,
$\{F'(\sigma,\tau)(i_p \otimes i_q)\}$ form a basis for $(S(X) \bigotimes S(Y))_n$(note that $F'(\sigma,\tau)(i_p \otimes i_q) = \sigma_\sharp \otimes \tau_\sharp(i_p \otimes i_q) = \sigma_\sharp(i_p) \otimes \tau_\sharp(i_q) = \sigma \circ i_p \otimes \tau \circ i_q = \sigma \otimes \tau$) since $\{\sigma \otimes \tau | \sigma \in S_p(X), \tau \in S_q(Y)\}$ form a basis for $S_p(X) \bigotimes S_q(Y)$ and hence $\{\sigma \otimes \tau | \sigma \in S_p(X), \tau \in S_q(Y), p+q=n\}$ form a basis for $(S(X) \bigotimes S(Y))_n$.

Proof of Eilenberg-Zilber theorem
Let $\tau_0 : H_0(F)(= H_0(X \times Y)) \leftrightarrow H_0(F')(= H_0(X) \bigotimes H_0(Y))$ be a natural transformation(isomorphism) determined by path components, i.e., if C, D are path-components of X, Y, respectively, then $C \times D$ is a path component of $X \times Y$ and $H_0(C \times D) \overset{\tau_0}{\leftrightarrow} H_0(C) \bigotimes H_0(D)$.
AMT $\Rightarrow \tau_0$ gives rise to a natural chain homotopy equivalence.

□

숙제 30 (29.14) Compute $H(\mathbb{P}^2 \times S^3) \neq H(\mathbb{P}^3 \times S^2)$ and show $\pi_*(\mathbb{P}^2 \times S^3) \cong \pi_*(\mathbb{P}^3 \times S^2)$.

숙제 31 (29.15) Compare $S^2 \times S^4$ and $\mathbb{C}P^3$. They both have same homology but π_4 are different.
(If necessary, use the following fibration and the corresponding long exact sequence of homotopy groups : $S^1 \to S^7 \to \mathbb{C}P^3 \Rightarrow \cdots \to \pi_k(S^1) \to \pi_k(S^7) \to \pi_k(\mathbb{C}P^3) \to \pi_{k-1}(S^1) \to \cdots$)

숙제 32 (29.11.2) $\chi(X \times Y) = \chi(X)\chi(Y)$.
(Use the Poincaré series of $X = f_X(t) := \beta_0 + \beta_1 t + \cdots + \beta_n t^n$ and notice that $\chi(X) = f_X(-1)$.)

13. Homology cross product

$$0 \to (H(X)\bigotimes H(Y))_n \xrightarrow{i} H_n(X\times Y) \longrightarrow (H(X) * H(Y))_{n-1} \to 0 \qquad \begin{array}{ccc} \{x\}\otimes\{y\} & \mapsto & \{x\}\times\{y\} \\ & \searrow & \uparrow \\ & & \{x\otimes y\} \end{array}$$

$$\begin{array}{ccccc} (H(X)\bigotimes H(Y))_n & \searrow & H_n(X\times Y) & \nearrow & (H(X) * H(Y))_{n-1} \\ & & \| & & \\ & & H_n(S(X)\bigotimes S(Y)) & & \end{array}$$

Note $\phi : S(X)\bigotimes S(Y) \to S(Y)\bigotimes S(X)$ given by $\phi(x\otimes y) = (-1)^{pq}y\otimes x$ is a natural chain map.
$\ulcorner \because \phi(\partial(x\otimes y)) = \phi(\partial x\otimes y + (-1)^p x\otimes \partial y) = (-1)^{(p-1)q} y\otimes \partial x + (-1)^p(-1)^{p(q-1)}\partial y \otimes x = \partial(\phi(x\otimes y)) \lrcorner$

$$\Rightarrow \quad \begin{array}{ccc} S(X\times Y) & \xrightarrow{T_\sharp} & S(Y\times X) \\ {\scriptstyle f_1}\downarrow\simeq & & {\scriptstyle f_2}\downarrow\simeq \\ S(X)\bigotimes S(Y) & \underset{\phi}{\overset{\simeq}{\to}} & S(Y)\bigotimes S(X) \end{array} \quad , \qquad T : X\times Y \to Y\times X \qquad T(x,y) = (y,x)$$

All the maps a natural chain transformations.
By the AMT, $T_\sharp \simeq$ "f_2^{-1}" $\circ\, \phi \circ f_1$.
$\Rightarrow T_*(\xi\times\eta) = (-1)^{pq}(\eta\times\xi), \xi\in H_p(X), \eta\in H_q(Y)$.

14. Alexander-Whitney diagonal approximation
We can give a specific chain map(and hence a chain homotopy equivalence.)

$A : S(X\times Y) \to S(X)\bigotimes S(Y)$ given by $\forall \omega \in S_n(X\times Y), \omega = (\sigma,\tau)$,
$A(\omega) = \bigoplus_{p=0}^{n} \sigma\lambda_p \otimes \tau\rho_{n-p} \in (S(X)\bigotimes S(Y))_n$.

(1) A is natural : only to check the commutativity of the following diagram.

$$\begin{array}{ccccc} S_n(X\times Y) & \xrightarrow{A} & (S(X)\bigotimes S(Y))_n & = & \bigoplus S_p(X)\bigotimes S_{n-p}(Y) \\ {\scriptstyle (f\times g)_\sharp}\downarrow & & \circlearrowleft & & \downarrow {\scriptstyle f_\sharp\otimes g_\sharp = \oplus f_{\sharp p}\otimes g_{\sharp n-p}} \\ S_n(X'\times Y') & \xrightarrow{A} & (S(X')\bigotimes S(Y'))_n & = & \bigoplus S_p(X')\bigotimes S_{n-p}(Y') \end{array}$$

Just note $f_\sharp(\sigma\lambda_p) = (f_\sharp\sigma)\lambda_p \Rightarrow !$.

(2) A is a chain map, i.e., $\partial A = A\partial$:
(Proof is essentially same as the proof of derivation property of cup product.)

$\therefore$ Eilenberg-Zilber Theorem $\Rightarrow A_* : H_*(X\times Y) \xrightarrow{\cong} H_*(S(X)\bigotimes S(Y))$

15. Relative Künneth formula

Künneth formula for $(X,A)\times(Y,B)$

Lemma (Relative Eilenberg-Zilber Theorem)
Suppose $\{X\times B, A\times Y\}$: excisive couple in $X\times Y$. Then
$S(X\times Y)/S(X\times B\cup A\times Y)\simeq S(X)/S(A)\bigotimes S(Y)/S(B)$ naturally.
Note that $S(X\times Y)/S(X\times B\cup A\times Y):=S((X,A)\times(Y,B)), S(X)/S(A)=S(X,A)$ and $S(Y)/S(B)=S(Y,B)$.
(Recall $\{A,B\}$ is excisive couple if $S(A)+S(B)\underset{\simeq}{\overset{i}{\hookrightarrow}} S(A\cup B)$.)

Proof
Excisive $\Rightarrow S(X\times B)+S(A\times Y)\overset{\simeq}{\hookrightarrow} S(X\times B\cup A\times Y)$
$\Rightarrow S(X\times Y)/(S(X\times B)+S(A\times Y))\overset{\simeq}{\hookrightarrow} S(X\times Y)/S(X\times B\cup A\times Y)$ naturally.

$$\begin{array}{ccc} S(A)\bigotimes S(Y) & \underset{\simeq}{\rightarrow} & S(A\times Y) \\ \cap & & \cap \\ S(X)\bigotimes S(Y) & \underset{\simeq}{\rightarrow} & S(X\times Y) \\ \cup & & \cup \\ S(X)\bigotimes S(B) & \underset{\simeq}{\rightarrow} & S(X\times B) \end{array}$$

EZ $\Rightarrow$ (the above) naturally.

$\Rightarrow$

$$\begin{array}{ccc} S(X\times Y)/(S(X\times B)+S(A\times Y)) & \underset{\simeq}{\leftarrow} & S(X)\bigotimes S(Y)/(S(X)\bigotimes S(B)+S(A)\bigotimes S(Y)) \\ \simeq\downarrow & & (*)\downarrow\cong \\ S(X\times Y)/S(X\times B\cup A\times Y) & & S(X)/S(A)\bigotimes S(Y)/S(B) \end{array}$$

$(*)$ follows from the general fact that$(X\bigotimes Y)/(X\bigotimes B+A\bigotimes Y)\overset{\cong}{\rightarrow} X/A\bigotimes Y/B$.
Indeed $\cong$ is induced from canonical map $X\times Y\rightarrow X/A\bigotimes Y/B$(cf 1.(3)).

□

From the lemma, we have the following relative Künneth formula.

Let R be a P.I.D. Then
$0\rightarrow (H(X,A)\bigotimes H(Y,B))_n \rightarrow H_n((X,A)\times(Y,B)) \rightarrow (H(X,A)*H(Y,B))_{n-1}\rightarrow 0$
, a natural short exact sequence which splits.
By definition, $H_n((X,A)\times(Y,B))=H_n(X\times Y, X\times B\cup A\times Y)$